On the relation of Manin's quantum plane and quantum Clifford algebras

Bertfried Fauser

Universität Konstanz, Fachbereich Physik, Fach M 678
78457 Konstanz, Germany
E-mail: Bertfried.Fauser@uni-konstanz.de

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One particular approach to quantum groups (matrix pseudo groups) provides the Manin quantum plane. Assuming an appropriate set of non-commuting variables spanning linearly a representation space one is able to show that the endomorphisms on that space preserving the non-commutative structure constitute a quantum group. The non-commutativity of these variables provide an example of non-commutative geometry.
In some recent work we have shown that quantum Clifford algebras –i.e. Clifford algebras of an arbitrary bilinear form– are closely related to deformed structures as q-spin groups, Hecke algebras, q-Young operators and deformed tensor products. The natural question of relating Manin's approach to quantum Clifford algebras is addressed here. Some peculiarities are outlined and explicite computations using the CLIFFORD Maple package are exhibited. The meaning of non-commutative geometry is re-examined and interpreted in Clifford algebraic terms.

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1 Introduction

The Manin quantum plane is one particular starting point to construct quantum groups or matrix pseudo groups [13, 18]. The main point in this construction is, that the non-commutativity of the "coordinates" induces a non-commutative tensor product. Since Clifford algebras provide an ideal tool to describe geometric settings and linear algebra [11] we try to give an outline in which way the Manin construction could be transfered into the Clifford framework. We found in former investigations already Hecke algebra representations [6] and q-spin groups [7] which motivated our work. However, the common notion of Clifford algebra has to be replaced by that of *quantum Clifford algebra*, that is a Clifford algebra of an arbitrary bilinear form, see [5]. Such type of quantum Clifford algebras do no longer possess decomposition theorems, but decompose in general only over deformed tensor products. Such deformations need not even to be braided. This results also in the fact that the Clifford functor is no longer exponential, but might be q-exponential in some cases. We examine the split case $C\ell_{p,p}$ for $p = 4$ in the undeformed and

deformed case and show that we can find x and y elements fulfilling the quantum plane relations. Also this investigation showed that it is fruitful to study the decomposition of Clifford algebras to learn about their composition.

2 The Manin quantum plane

In this section we give a short account on the (two dimensional) quantum plane as constructed by Manin [13]. We fix thereby our notation.

Let V be a linear space over a field $\mathbb{k}$. The dimension of V is given as the number of *linear* independent elements. Consider the space $\otimes^2 V$ of tensors of degree two and impose on elements x and y which span V, i.e. $V = < x, y >$, the condition

$$x \otimes y = q\, y \otimes x\,. \tag{1}$$

Usually the tensor-product is not explicitely written. In a second step, one considers the endomorphisms of V, such that the condition (1) remains invariant ($M \in \mathrm{End}(V)$; $\vec{x}, \vec{x}' \in V$)

$$M\vec{x} = \vec{x}' \tag{2}$$

i.e. one requires that

$$x' \otimes y' = q\, y' \otimes x'\,. \tag{3}$$

The relations imposed on the *quantum plane* V result in relations of the matrix elements of M w.r.t. a basis of V and V^*. Such matrix elements are necessarily non-commutative. Obviously one has $V^* =< \partial_x, \partial_y >$ and the dual quantum plane fulfils the relation

$$\partial_x \otimes \partial_y = q^{-1}\, \partial_y \otimes \partial_x. \tag{4}$$

This space constitutes the canonical dual space of V. Deducing the corresponding commutation relations one obtains

$$\begin{aligned}
M \quad &\cong \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad\qquad (5)\\
ab \;=\; q^{-1}\, ba \quad & cd \;=\; q^{-1}\, dc \\
ac \;=\; q^{-1}\, ca \quad & bd \;=\; q^{-1}\, db \\
bc = cb \quad & ad - da = (q^{-1} - q)\, bc.
\end{aligned}$$

Finally one can define a trace and a quantum determinant. The later is given as $\mathrm{Det}_q(M) = ad - q^{-1}bc$ and is a central element of the $GL_q(2)$.

For our purpose it is important to note, that quantum endomorphisms can be realized as matrices with matrix entries which fulfil some constraints. Such block matrices constitute then a *matrix pseudo group* [18], since the set of matrices is not a full matrix algebra but $\mathrm{Mat}_{n\times n}$ mod relations.

3 Composition and decomposition

A main result in the theory of real and complex Clifford algebras is the occurrence of certain periodicity theorems [15, 12, 2]. Albert Crumeyrolle took them as the Mendeljev periodic system of elementary particles [14], page X: *les théorèmes de périodicité pour les algèbres de Clifford devraient jouer, pour la théorie des particules élémentaires, un rôle analogue à celui joué par la classification périodique des éléments de Mendeleïev en Chimie*. However, such periodicity theorems break down for quantum Clifford algebras.

3.1 Periodicity theorems I:

Let $\mathsf{H} = (V, Q)$ be a quadratic space, that is a pair of a $\mathbb{k}$-linear space V and a quadratic form Q and **Quad** the corresponding category. If char $\mathbb{k} \neq 2$ then Q has a symmetric matrix representation. There is a functorial mapping of quadratic spaces called $C\ell$ into the associative unital algebras

$$\mathbf{Quad} \xrightarrow{C\ell} \mathbf{Alg}. \tag{6}$$

The particular map $\gamma : (V, Q) \mapsto C\ell(V, Q)$ is called Clifford map [3]. The (universal) algebra connected in this way to a quadratic space is called Clifford algebra. From the basic law, that $V \ni x \mapsto \gamma_x$ and $\gamma_x \gamma_x = Q(x)\mathbb{1}$ one obtains by polarization the common anti-commutation relations

$$\begin{aligned} i) \quad & \gamma_x\gamma_y + \gamma_y\gamma_x = 2g(x, y) \\ ii) \quad & Q(x + y) - Q(X) - Q(y) =: 2g(x, y). \end{aligned} \tag{7}$$

During the process of classifying such algebras one obtains periodicity theorems, as already Clifford proved some of them. In particular one has

$$\begin{aligned} i) \quad C\ell_{p,q+8} &\cong C\ell_{p,q} \otimes \mathbb{R}(16) \\ ii) \quad C\ell_{p+1,q+1} &\cong C\ell_{p,q} \otimes C\ell_{1,1} \\ &\cong C\ell_{p,q} \otimes \mathbb{R}(2) \end{aligned} \tag{8}$$

etc., where p and q are sufficient large to allow the decomposition. $C\ell_{p,q}$ is the real Clifford algebra $C\ell(V, Q)$ with a quadratic form Q of signature p, q and $\mathbb{R}(n)$ is the *full* $n \times n$ matrix algebra over the reals. Obviously one can reduce higher dimensional Clifford algebras by splitting them into smaller parts. As a consequence of this process one stays with a few basic irreducible (smallest or elementary) Clifford algebras. This process can be reversed to *compose* any real Clifford algebra from such building blocks. A particular interesting case for our purpose is (8-ii) where we can look at $C\ell_{p+1,q+1}$ as a 2×2 matrix algebra over the non-commutative ring $C\ell_{p,q}$. Elements $M \in C\ell_{p+1,q+1}$ can be written as a 2×2 block matrix containing $m_{ij} \in C\ell_{p,q}$ as entries, where the indices i, j are w.r.t. $\mathbb{R}(2)$. At this stage it is important that the classification of Clifford algebras is identical to the classification of the involved quadratic forms Q.

We can learn from the decomposition theorems moreover, that the Clifford functor $C\ell$ is exponential. That is a direct sum of two spaces is mapped onto the tensor-product of Clifford algebras

$$C\ell(U \oplus V, Q_1 \perp Q_2) \cong C\ell(U, Q_1) \otimes C\ell(V, Q_2). \tag{9}$$

3.2 Quantum Clifford algebras

Let $\mathsf{R} = (V, B)$ be a reflexive space, that is a $\mathbb{k}$-linear space V and a bilinear not necessarily symmetric form $B : V \times V \mapsto \mathbb{k}$. To define the associated *quantum Clifford algebra* we start defining the exterior algebra $\bigwedge V$ over V,

$$\bigwedge V = \mathbb{k} \oplus V \oplus \wedge^2 V \oplus \ldots \oplus \wedge^n V \oplus \ldots . \tag{10}$$

We introduce the dual space V^* of linear forms on V and a so-called Euclidean dual isomorphism $* : V \mapsto V^*$ to introduce a dual basis as a particular isomorphism $V^* \ni i_x \cong x^*$. Let $x, y \in V$ and $x^* \in V^*$ the image of x in V^*. The contraction is defined as

$$i_x(y) \;=\; x^*(y) = x \underset{B}{\rfloor} y \;=\; B(x, y). \tag{11}$$

We write $\rfloor$ for short if it is clear which bilinear form is involved in the definition of the contraction. Following the ideas of Chevalley [3] we note that with $x \in V$

$$\gamma_x = x \underset{B}{\rfloor} \; + x\wedge \tag{12}$$

is a Clifford map of $\mathsf{R} = (V, B)$ into **Alg**. The particular Clifford algebra appears as strict subalgebra of the endomorphism algebra $C\ell(V, B) \subset \mathrm{End}(\bigwedge V)$ of the exterior algebra $\bigwedge V$. The action of γ_x –we use x for short from now on– induces an action on the whole exterior algebra demanding that ($x, y \in V$; $u, v, w \in \bigwedge V$)

$$\begin{aligned} &i) && x \rfloor_B y &&= B(x, y) \\ &ii) && x \rfloor_B (u \wedge v) &&= (x \underset{B}{\rfloor} u) \wedge v + \hat{v} \wedge (x \underset{B}{\rfloor} u) \\ &iii) && (u \wedge v) \rfloor_B w &&= u \underset{B}{\rfloor} (v \underset{B}{\rfloor} w), \end{aligned} \tag{13}$$

where $\hat{v}$ is the Grassmann grade involution defined as $\hat{v} = (-1)^{\partial v} v$, and ∂v is the Grassmann grade of the element v. See [4] for an Hopf algebraic explanation of the grade involution as the antipode.

Quantum Clifford algebras are denoted as $C\ell(V, B)$ they have been investigated in [6, 8, 5]. As a matter of fact we lack a classification of this type of algebras.

3.3 Periodicity theorems II:

We have the following theorem [5]:
Let $C\ell(V, Q)$ and $C\ell(V, B)$ be two Clifford algebras over the space V such that

$Q(x) = B(x,x)$, i.e. Q and B describe the same quadratic form, then there exists a unique $\mathbb{Z}_2$-graded Wick-isomorphism ϕ connecting the algebras via

$$\phi \circ C\ell(V,Q) = C\ell(V,B). \tag{14}$$

This theorem can be used to induce periodicity theorems on $C\ell(V,B)$. However, this can be achieved *only* if we allow *deformed* tensor products in general. Let $W = V \perp_Q U = V \oplus U$ furthermore $Q_1 := Q|_V$, $Q_2 = Q|_U$ the restrictions of Q on V and U, then we decompose $C\ell(V,B)$ as

$$\begin{aligned} C\ell(W,B) &= \phi \circ C\ell(W,Q) \;=\; \phi \circ C\ell(V \oplus U, Q_1 \oplus Q_2) \\ &= \phi \circ (C\ell(V,Q_1) \otimes C\ell(V,Q_2)) \\ &= (\phi|_V \circ C\ell(V,Q_1)) \otimes|_\phi (\phi|_U \circ C\ell(U,Q_2)) \\ &= C\ell(V,B_1) \otimes|_\phi C\ell(U,B_2). \end{aligned} \tag{15}$$

But in general ϕ does *not* factorize w.r.t. Q, as indicated by the ϕ-deformed tensor product. We note that the Clifford functor is *not* exponential for quantum Clifford algebras in general. There might be a chance to find ϕ-deformed exponentials. However, the above defined deformation is not e.g. braided by construction but the most general possible deformation. This follows from the observation that we have no further freedom in the reflexive space R and that the Clifford functor is injective.

4 Reconstruction of the quantum plane

4.1 The matrix window

We know from the structure theory of real Clifford algebras that such algebras possess faithful irreducible representations on spinor spaces over distinguished rings. Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, {}^2\mathbb{R}, {}^2\mathbb{H}\}$, then one obtains representations ρ with

$$\rho(C\ell_{p,q}) \cong \mathrm{Mat}_{\mathbb{L}}(n \times n), \tag{16}$$

where $\mathbb{L}$ is chosen accordingly to the Radon-Hurwitz number. It is important to note that such representations map Clifford algebras onto *full* matrix algebras and that there are no further relations in force.

Using tensor products of matrices, one has to give an identification of bases of tensor factors. This means that one has no *a priori* information about the relation of bases of different tensor factors. This is especially important if one wants to switch entries in a tensor product, e.g. ($x_{\{e\}}$ is x in basis $\{e\}$)

$$sw(x_{\{e\}} \otimes y_{\{f\}}) = y_{\{e\}} \otimes x_{\{f\}}. \tag{17}$$

The Clifford approach is intrinsically invariant and basis free. Furthermore, if one wants to *compose* Clifford algebras, one can learn about this construction by *decomposing* larger Clifford algebras into factors.

It is important to note that only spinor bases provide representations of Clifford algebras as full matrix algebras. Representation indices are spinor indices.

Spinor spaces can be constructed inside the Clifford algebra as minimal left (right) ideals generated from primitive (indecomposable) idempotent elements f_{ii}. One finds

$$\begin{aligned} \mathcal{S} &\cong C\ell f_{11} \\ \{e\} &= \{f_{11}, x_2 f_{11}, \dots, x_n f_{11}\} \;=\; \{f_{11}, f_{21}, \dots, f_{n1}\} \end{aligned} \tag{18}$$

where the spinor basis $\{e\}$ is generated by f_{11} and the x_i which intertwine the different primitive idempotent elements of the Clifford algebra

$$x_r f_{11} = f_{rr} x_r \tag{19}$$

in a compatible way, i.e. $x_r = x_s x_t$ if $s,t < r$ if possible. Such bases can be found e.g. in [9].

4.2 Computational examples

4.2.1 Undeformed case:

The space time split [17] as the modulo $(1,1)$ periodicity [12] –called also conformal split w.r.t. $C\ell_{1,1}$– have been extensively studied in [11]. We restrict ourself to the split Clifford algebras $C\ell_{p,p}$ and $p=2$. From (8) we have

$$\begin{aligned} C\ell_{2,2} &\cong C\ell_{1,1} \otimes C\ell_{1,1} \;\cong\; C\ell_{1,1} \otimes \mathbb{R}(2) \\ C\ell_{2,2} &\ni M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad a,b,c,d \in C\ell_{1,1}\,. \end{aligned} \tag{20}$$

Let $\{e_i\}$ and $\{f_j\}$ be the generators of the first and second $C\ell_{1,1}$ algebras. Both of them are isomorphic to $\mathbb{R}(2)$ and possess matrix bases E_{ij} and F_{ij}. In a spinor basis ξ_α this is $(E_{ij})_{\alpha\alpha'} = \delta_{i\alpha}\delta_{j\alpha'}$. Any element of $C\ell_{2,2}$ can be written as

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; aF_{11} + bF_{12} + cF_{21} + dF_{22}. \tag{21}$$

It can be checked, that $\{e_i \otimes f_1 f_2\}$ and $\{\mathbb{1} \otimes f_j\}$ generate $C\ell_{2,2}$.

The **spin** and **pin** groups of Clifford algebras are defined to be sub groups of the Clifford Lipschitz group $\Gamma(p,q)$, which is the subgroup of multiplicative units of the Clifford algebra, denoted as $C\ell^*$, preserving the $\mathbb{Z}_n$-multi-vector grading

$$\Gamma(p,q) := \{g \in C\ell^*_{p,q} \;:\; gVg^{-1} \in V\}. \tag{22}$$

Using the reversion ($(AB)\tilde{} = A\tilde{}B\tilde{}$, $e_i\tilde{} = e_i$, $\mathbb{1}\tilde{} = \mathbb{1}$) and the grade involution $\hat{}\, : V \mapsto -V$ as defined in (13) one composes two anti-involutions α_ϵ as $\alpha_1 = \tilde{}$ and $\alpha_{-1} = \hat{} \circ \tilde{}$. Define

$$\begin{aligned} \mathbf{pin}(p,q) &:= \{g \in \Gamma(p,q) \;:\; g\alpha_{-1}(g) = \pm 1\} \\ \mathbf{spin}(p,q) &:= \mathbf{pin} \cap C\ell^+(p,q), \end{aligned} \tag{23}$$

where $C\ell^+$ is the even part of $C\ell$ w.r.t. the $\mathbb{Z}_2$-grading. The elements of the Clifford–Lipschitz group are called *versors* and are decomposable into Clifford products of vectors –i.e. grade 1– elements of $C\ell$.

Conformal transformations have been investigated by Vahlen [16]. In Clifford terms one is interested in elements M from $C\ell_{1,1} \otimes \mathrm{I\!R}(2)$ which respect the $\mathbb{Z}_4$-grade of $C\ell_{2,2}$. This imposes certain restrictions on the matrix elements of M. One obtains [12]:

$$
\begin{aligned}
i)&\quad a\tilde{a}, b\tilde{b}, c\tilde{c}, d\tilde{d} \in \mathrm{I\!R} \\
ii)&\quad a\tilde{c}, b\tilde{d} \in V \\
iii)&\quad av\tilde{b} + b\tilde{v}\tilde{a}, cv\tilde{d} + d\tilde{v}\tilde{c} \in \mathrm{I\!R} \quad \forall\, v \in V \\
iv)&\quad av\tilde{d} + b\tilde{v}\tilde{c} \in V \quad \forall\, v \in V \\
v)&\quad a\hat{\tilde{b}} = b\hat{\tilde{a}}, \quad c\hat{\tilde{d}} = d\hat{\tilde{c}} \\
vi)&\quad a\hat{\tilde{d}} - b\hat{\tilde{c}} = \pm 1.
\end{aligned}
\tag{24}
$$

The last two conditions are normalization conditions and $vi)$ could be called pseudo determinant. Compare this with the relations (5) and the q-determinant there. This type of matrices is suitable to model conformal geometry.

4.2.2 Deformed case:

We use CLIFFORD, a Maple V package developed by Rafał Abłamowicz [1] to demonstrate that we can indeed find nontrivial elements fulfilling the Manin quantum plane relations. Therefore we search for two elements x and y in $C\ell(V, B)$, where B is defined to be

$$
B := \begin{pmatrix} g_{11} & g_{12} + A_{12} \\ g_{12} - A_{12} & g_{22} \end{pmatrix}. \tag{25}
$$

A spinor basis consists of an idempotent element and a second linear independent element. Starting with arbitrary elements, CLIFFORD finds after having set an arbitrary parameter to zero (giving results only, because lack of space)

$$
\begin{aligned}
x &:= \frac{1}{2}(1 + A_{12})\mathrm{1\!I} + \frac{\sqrt{g_{11}(\det(g) + 1)}}{2g_{11}}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_1 \wedge \mathbf{e}_2 \\
y &:= \frac{1}{2}\sqrt{g_{11}(\det(g) + 1)}\mathrm{1\!I} + \frac{1}{2}(1 - g_{12})\mathbf{e}_2 + \frac{1}{2}g_{11}\mathbf{e}_1,
\end{aligned}
\tag{26}
$$

where $\det(g)$ is the determinant of the symmetric part of B. After solving the $RXY = YXR$ equation, we find two linearly independent solutions for R

$$
\begin{aligned}
R_1 &:= (A_{12} - 1)\mathrm{1\!I} + \frac{\sqrt{(\det(g) + 1)g_{11}}}{g_{11}}\mathbf{e}_1 + \mathbf{e}_1 \wedge \mathbf{e}_2 \\
R_2 &:= \frac{\sqrt{(\det(g) + 1)g_{11}}}{g_{11}}\mathrm{1\!I} + \frac{1 - g_{12}}{g_{11}}\mathbf{e}_1 + \mathbf{e}_2.
\end{aligned}
\tag{27}
$$

Obviously we could also find q, however as an algebra element, which relates the two basis elements x and y. From this formulas, it is clear that the antisymmetric part is essential to be able to calculate such R_i elements. The term quantum Clifford algebra was chosen to exhibit this connection to deformed structures in the case of non-symmetric bilinear forms.

The above discussed Vahlen matrices of conformal transformations will be affected by such a deformation too. One would expect to find as a special case q-deformed conformal transformations.

In Hahn [10] it was shown that Clifford algebras have a separability idempotent element which allows one to inject the algebra, here $C\ell_{1,1}$ into its enveloping algebra, here $C\ell_{2,2} \cong C\ell_{1,1} \otimes C\ell_{1,1}$. The former algebra is a bi-module under the action of the enveloping algebra and the unit is the inverse image of the separability idempotent. This structure defines in fact the (undeformed) co-product. The deformed case of this structure will be investigated in detail elsewhere.

5 conclusion

We showed that quantum Clifford algebras can be used to investigate deformed tensor products and the groups acting on modules over such deformed spaces. Examining the decomposition of ordinary and quantum Clifford algebras, we have been able to show that quantum plane relations can be modelled in this framework. This supports our finding of q-spin groups and Hecke algebra representations in previous works. Vahlen matrices and the conformal split seem to be the undeformed counterparts of the quantum plane. This sheds some light on the geometrical meaning of quantum planes and non-commutative geometry, which in our opinion is a symmetry of "composed" objects. In geometrical terms we have introduced inhomogeneous coordinates and a metric by introducing a Cayley-Klein type measure induced by the factorization [17]. An important issue for further research is the connection of quantum Clifford algebras and Hopf algebras as examined in [4]. A more satisfying theory would definitively incorporate the Hopf algebra structure and develop all groups and invariants from that starting point.

References

[1] R. Abłamowicz: in *Clifford (Geometric) Algebras*, (Ed. W.E. Baylis), Birkhäuser, Boston, 1996, p. 167.
available at `http://math.tntech.edu/rafal/cliff4/`

[2] P. Budinich, A. Trautman: *The Spinorial Chessboard*, Trieste Notes in Physics, Springer, Berlin, 1988.

[3] C. Chevalley: *The Algebraic Theory of Spinors*, Columbia University Press, New York, 1954.

[4] B. Fauser: *On the Hopf algebraic origin of Wick normal-ordering*, submitted, hep-th/0007032.

[5] B. Fauser, R. Abłamowicz: in *Clifford Algebras and their Applications in Mathematical Physics*, (Eds. R. Abłamowicz, B. Fauser), Birkhäuser, Boston, 2000, p.347.

[6] B. Fauser: J. Phys. A: Math. Gen. **32**, 1999, p. 1919.

[7] B. Fauser: in *GROUP22, XXII Int. Colloquium on Grp. Theor. Meth. in Phys.*, (Eds. S.P. Corney, R. Delbourgo, P.D. Jarvis), International Press, Cambridge Ma., 1999, p. 413.

[8] B. Fauser: J. Math. Phys. **39**, 1998, p. 4928.

[9] B. Fauser: in *Clifford algebras and their Application in Mathematical Physics*, (Eds. V. Dietrich, K. Habetha, G. Jank), Kluwer, Dordrecht, 1998, p. 89.

[10] A.J. Hahn: *Quadratic Algebras, Clifford Algebras, and Arithmetic Witt Groups*, Springer, New York, 1994.

[11] D. Hestenes: Acta Appl. Math. 23, 1991, p. 65.

[12] J. Maks: *Modulo (1,1) periodicity of Clifford algebras and the generalized (anti)-Möbius transformation*, Thesis, TU Delft, 1989.

[13] Y. Manin: *Quantum Groups and Non-Commutative Geometry*, Centre de Recherches Mathématiques, Université de Montréal, 1988

[14] A. Micalli: *Albert Crumeyrolle, la démarche algébrique d'un Géomètre* in Clifford Algebras and Spinor Structures, (Eds. R. Abłamowicz, P. Lounesto), Kluwer, Dordrecht 1995, p. ix.

[15] I. Porteous: *Topological Geometry*, Van Nostrand, New York, 1969.

[16] K.Th. Vahlen: Math. Ann. 55, 1902, p. 585.

[17] H. Weyl: *Raum – Zeit – Materie*, Springer, Heidelberg, 7th ed. 1988.

[18] S. L. Woronowicz: Commun. Math. Phys. 111, 1987, p. 613; ibid. 122, 1989, p. 125.